Ramsey theory and topological dynamics for first order theories, and an abstract generalization

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General look on the subject

Kechris, Pestov, Todorčević (or KPT) theory

Ramsey-theoretic properties of a Fraïssé class $\leftrightsquigarrow$ dynamical properties of the automorphism group of the Fraïssé limit

The theory developed in [KLM]

"Definable" Ramsey-theoretic properties of a first order theory T $\leftrightsquigarrow$ dynamical properties of T

This leads to combinatorial criteria for profiniteness and for triviality of the Ellis group of the theory T and in consequence for equality of Kim-Pillay and Shelah strong types (which is fundamental in model theory).

[KLM] K. Krupiński, J. Lee, S. Moconja, *Ramsey theory and topological dynamics for first order theories*, Trans. Amer. Math. Soc. 375 (2022), 2553-2596.

An abstract generalization by Krupiński and Lee

"Definable" Ramsey-theoretic properties of a zero-dimensional ambit $(G, X, x_0) \leftrightsquigarrow$ dynamical properties of (G, X, x_0)

This also leads to combinatorial criteria for profiniteness and for triviality of the Ellis group of (G, X).

Three important specializations to the following ambits

1. $(\mathrm{Aut}(\mathfrak{C}), S_{\bar{c}}(\mathfrak{C}), \mathrm{tp}(\bar{c}/\mathfrak{C}))$, where $\mathfrak{C}$ is a monster model of a theory T, $\bar{c}$ an enumeration of $\mathfrak{C}$, and $S_{\bar{c}}(\mathfrak{C}) := \{p \in S_{\bar{x}}(\mathfrak{C}) : \mathrm{tp}(\bar{c}/\emptyset) \subseteq p\}$.
2. $(G, S_G(M), \mathrm{tp}(e/M))$, where G is a group definable in a structure M.
3. The universal ambit Σ^M of the topological group $\mathrm{Aut}(M)$ of the automorphisms of a structure M.

Topological dynamics

Let (G, X) be a flow. For every $g \in G$ we have $\pi_g : X \to X$ given by $\pi_g(x) := gx$.

Definition/Fact

$E(X)$ is defined as the closure of $\{\pi_g : g \in G\} \subseteq X^X$ in the pointwise convergence topology on X^X. Then $E(X)$ with $\circ$ (i.e. composition) is a left topological semigroup which is compact. It is called the *Ellis semigroup* of the flow (G, X).

Definition

The *Ellis group* of the flow (G, X) is the isomorphism type of the isomorphic groups of the form $u\mathcal{M}$ for any minimal left ideal $\mathcal{M}$ of $E(X)$ and $u \in J(\mathcal{M}) := \{v \in \mathcal{M} : v^2 = v\}$. Any group $u\mathcal{M}$ as above will also be called the Ellis group of X.

The Ellis group is equipped with the τ*-topology* which is quasi-compact, T_1, and the group operation is separately continuous.

Abstract context

From now on, (G, X, x_0) is a 0-dimensional ambit and $\mathcal{B}$ the family of all clopen subsets of X.

Definition of definable colorings (KL)

Fix any $S \subseteq X$. A coloring $c\colon G \to 2^n$ is called *S-definable* if there are $U_0, \ldots, U_{n-1} \in \mathcal{B}$ and $y_0, \ldots, y_{n-1} \in S$ such that

$$c(g)(i) = \begin{cases} 1, & \text{if } y_i \in gU_i \\ 0, & \text{if } y_i \notin gU_i \end{cases}$$

for any $g \in G$ and $i < n$. A coloring $c\colon G \to 2^n$ is

- *strongly definable* if it is $\{x_0\}$-definable,
- *definable* if it is Gx_0-definable,
- *externally definable* if it is X-definable.

Abstract context — cont.

Definition of definable Ramsey properties (KL)

Fix any $S \subseteq X$. The ambit (G, X, x_0) has *S-definable embedding Ramsey property* (S-DERP) if for every S-definable coloring $c\colon G \to 2^n$ and finite $A \subseteq G$ there is $h \in G$ such that $|c[hA]| = 1$.

Definition of definable Ramsey properties — cont. (KL)

The ambit (G, X, x_0) has

- *weak definable embedding Ramsey property* (WDERP) if it has $\{x_0\}$-DERP,
- *definable embedding Ramsey property* (DERP) if it has Gx_0-DERP,
- *externally definable embedding Ramsey property* (EDERP) if it has X-DERP.

Abstract context — cont.

Theorem (KL)

Let $S \subseteq X$. TFAE

1. The ambit (G, X, x_0) has S-DERP.
2. There exists $\eta \in E(X)$ such that $\eta[S] \subseteq \text{Inv}(X) := \{x \in X : gx = x \text{ for all } g \in G\}$.

Corollary

1. (G, X, x_0) has WDERP iff $\text{Inv}(X) \neq \emptyset$.
2. (G, X, x_0) has DERP iff there exists $\eta \in E(X)$ with $\eta[Gx_0] \subseteq \text{Inv}(X)$.
3. (G, X, x_0) has EDERP iff there exists $\eta \in E(X)$ with $\text{Im}(\eta) \subseteq \text{Inv}(X)$.

Corollary

If (G, X, x_0) has EDERP, then any minimal left ideal of $E(X)$ is trivial, and so is the Ellis group of (G, X).

Abstract context — cont.

Let $G\backslash\mathcal{B} := \{GU : U \in \mathcal{B}\}$. For $\Delta \subseteq G\backslash\mathcal{B}$, let $B(\Delta)$ be the Boolean algebra generated by $\bigcup\Delta$, and $S_\Delta(X)$ the space of ultrafilters on $B(\Delta)$; $S(X)$ denotes the space of ultrafilters on $\mathcal{B}$.

Remark

$X \cong S(X) \cong \varprojlim S_\Delta(X)$, where Δ ranges over the finite subsets of $G\backslash\mathcal{B}$.

Proposition (KL)

TFAE and imply profiniteness of the Ellis group of X.

1. $X \cong \varprojlim X_i$, where the Ellis groups of all the X_i's are finite.
2. The Ellis group of each $S_\Delta(X)$ (where Δ ranges over the finite subsets of $G\backslash\mathcal{B}$) is finite.

Corollary

If for every finite $\Delta \subseteq G\backslash\mathcal{B}$ there exists $\eta \in E(S_\Delta(X))$ with finite image, then the Ellis group of X is profinite.

Abstract context — cont.

Definition (KL)

Fix a finite $\Sigma \subseteq \mathcal{B}$. A coloring $c\colon G \to 2^n$ will be called an *externally definable Σ-coloring* if external definability of c is witnessed by $U_0, \dots, U_{n-1} \in \Sigma$.

Definition (KL)

(G, X, x_0) has *separately finite externally definable embedding Ramsey degree* (sep. fin. EDERdeg) if for every finite $\Sigma \subseteq \mathcal{B}$ there exists $l \geq 1$ such that for every n and every externally definable Σ-coloring $c\colon G \to 2^n$, for any finite $A \subseteq G$ there exists $h \in G$ with $|c[hA]| \leq l$.

Abstract context — cont.

Theorem (KL)

TFAE

1. (G, X, x_0) has sep. finite EDERdeg.
2. For every finite $\Delta \subseteq G \backslash \mathcal{B}$ there exists $\eta \in E(S_\Delta(X))$ with finite image.

Corollary

If (G, X, x_0) has sep. finite EDERdeg, then the Ellis group of X is profinite, which in turn implies that any minimal distal flow being a homomorphic image of the flow (G, X) is a profinite flow.

First order theories

Let T be a theory, and $\mathfrak{C} \models T$ a monster model. For a finite tuple $\bar{a} \subseteq \mathfrak{C}$

$$\binom{\mathfrak{C}}{\bar{a}} = \{\bar{a}' \in \mathfrak{C}^{|\bar{a}|} : \bar{a}' \equiv \bar{a}\}.$$

Definition (KLM)

A coloring $c : \binom{\mathfrak{C}}{\bar{a}} \to 2^n$ is *definable* if there are formulas $\varphi_i(\bar{x}, \bar{b})$, $i < n$, with parameters $\bar{b}$ from $\mathfrak{C}$, such that:

$$c(\bar{a}')(i) = \begin{cases} 1, & \text{if } \models \varphi_i(\bar{a}', \bar{b}) \ \ (\text{iff } \varphi_i(\bar{a}', \bar{y}) \in \mathrm{tp}(\bar{b}/\mathfrak{C})) \\ 0, & \text{if } \models \neg\varphi_i(\bar{a}', \bar{b}) \ \ (\text{iff } \neg\varphi_i(\bar{a}', \bar{y}) \in \mathrm{tp}(\bar{b}/\mathfrak{C})) \end{cases}$$

for any $\bar{a}' \in \binom{\mathfrak{C}}{\bar{a}}$ and $i < n$.

First order theories — cont.

Definition (KLM)

A coloring $c\colon \binom{\mathfrak{C}}{\bar{a}} \to 2^n$ is *externally definable* if there are formulas $\varphi_0(\bar{x},\bar{y}),\dots,\varphi_{n-1}(\bar{x},\bar{y})$ without parameters and types $p_0(\bar{y}),\dots,p_{n-1}(\bar{y}) \in S_{\bar{y}}(\mathfrak{C})$ such that:

$$c(\bar{a}')(i) = \begin{cases} 1, & \varphi_i(\bar{a}',\bar{y}) \in p_i(\bar{y}) \\ 0, & \neg\varphi_i(\bar{a}',\bar{y}) \in p_i(\bar{y}) \end{cases}$$

for any $\bar{a}' \in \binom{\mathfrak{C}}{\bar{a}}$ and $i < n$.

Remark

A coloring $c : \binom{\mathfrak{C}}{\bar{a}} \to 2^n$ is definable iff it is externally definable via realized (in $\mathfrak{C}$) types $p_0(\bar{y}),\dots,p_{n-1}(\bar{y}) \in S_{\bar{y}}(\mathfrak{C})$.

Remark

In the definition of externally definable colorings, wlog $\bar{y}$ corresponds to $\bar{c}$ and $p_0(\bar{y}) = \cdots = p_{n-1}(\bar{y}) \in S_{\bar{c}}(\mathfrak{C})$.

First order theories — cont.

Definition (KLM)

1. T has EERP (the *elementary embedding Ramsey property*) if for any two finite tuples $\bar{a} \subseteq \bar{b} \subseteq \mathfrak{C}$, any $n < \omega$, and any coloring $c : \binom{\mathfrak{C}}{\bar{a}} \to 2^n$ there exists $\bar{b}' \in \binom{\mathfrak{C}}{\bar{b}}$ such that $\binom{\bar{b}'}{\bar{a}}$ is monochromatic with respect to c.
2. T has DEERP (the *definable elementary embedding Ramsey property*) if the same holds but only for *definable* colorings c.
3. T has EDEERP (the *externally definable elementary embedding Ramsey property*) if the same holds but only for *externally definable* colorings c.

Remark (KLM)

1. These definitions do not depend on the choice of the monster (or just $\aleph_0$-saturated) model $\mathfrak{C}$.
2. EERP $\Longrightarrow$ EDEERP $\Longrightarrow$ DEERP.

First order theories — cont.

Remark

The theory of any $\aleph_0$-saturated Fraïssé structure with the embedding Ramsey property satisfies EERP. There are plenty of such examples in structural Ramsey theory.

Example (KLM)

Stable theories with $\operatorname{acl}^{eq}(\emptyset) = \operatorname{dcl}^{eq}(\emptyset)$ have EDEERP. For example, $T := ACF_0$ with named constants from the algebraic closure of $\mathbb{Q}$ has EDEERP, but not EERP.

Example (KLM)

The theory of the random n-hypergraph or random $(n, n+1)$-hypergraph has EDEERP.

Example (KLM)

The theory of the random $(2, 4)$-hypergraph has DEERP but not EDEERP.

First order theories — cont.

Remark

T has [E]DEERP iff $(\mathrm{Aut}(\mathfrak{C}), S_{\bar{c}}(\mathfrak{C}), \mathrm{tp}(\bar{c}/\mathfrak{C}))$ has [E]DERP.

Reason: $c\colon \binom{\mathfrak{C}}{\bar{a}} \to 2^n \rightsquigarrow c'\colon \mathrm{Aut}(\mathfrak{C}) \to 2^n$ given by
$c'(\sigma)(i) := c(\sigma(\bar{a}))(i)$
$c'\colon \mathrm{Aut}(\mathfrak{C}) \to 2^n \rightsquigarrow c\colon \binom{\mathfrak{C}}{\bar{a}} \to 2^n$ given by $c(\sigma(\bar{a}))(i) := c'(\sigma)(i)$,
where $\bar{a}$ is a tuple of parameters in the formulas yielding clopens from the definition of c'

Corollary (KLM)

1. T has DEERP iff T is extremely amenable (i.e., there exists an invariant type in $S_{\bar{c}}(\mathfrak{C})$).
2. T has EDEERP iff there exists $\eta \in E(S_{\bar{c}}(\mathfrak{C}))$ with $\mathrm{Im}(\eta) \subseteq \mathrm{Inv}_{\bar{c}}(\mathfrak{C}) := \{p \in S_{\bar{c}}(\mathfrak{C}) : p \text{ invariant}\}$.
3. If T has EDEERP, then the Ellis group of T (i.e., of $(\mathrm{Aut}(\mathfrak{C}), S_{\bar{c}}(\mathfrak{C}))$) is trivial, and so Lascar, Kim-Pillay, and Shelah strong types coincide.

First order theories — cont.

Similarly, the notion of sep. fin. EDERdeg applied to the ambit $(\mathrm{Aut}(\mathfrak{C}), S_{\bar{c}}(\mathfrak{C}), \mathrm{tp}(\bar{c}/\mathfrak{C}))$ yields the notion of T having *sep.fin. EDEERdeg* introduced in [KLM], and the characterization stated on slide 10 yields the main result of [KLM].

Theorem (KLM)

T has sep. fin. EDEERdeg iff for every finite set of formulas $\Delta(\bar{x}, \bar{y})$ (where $\bar{x}$ corresponds to $\bar{c}$) and finite set P of types in $S_{\bar{y}}(\emptyset)$ there exists $\eta \in E(S_{\bar{c},\Delta}(P))$ with finite image, where $S_{\bar{c},\Delta}(P)$ is the $\mathrm{Aut}(\mathfrak{C})$-flow of all maximal types consistent with $\mathrm{tp}(\bar{c}/\emptyset)$ which consist of formulas $\delta(\bar{x}, \bar{b})$ and their negations for δ ranging over Δ and $\bar{b}$ over the realizations of types from P.

Corollary (KLM)

Every theory with sep. fin. EDEERdeg has profinite Ellis group, and so Kim-Pillay and Shelah strong types coincide.

First order theories — cont.

Remark

The theory of any $\aleph_0$-saturated Fraïssé structure with the finite embedding Ramsey degree has sep. fin. EDEERdeg. There are plenty of such examples in structural Ramsey theory.

Example (KLM)

Stable theories have sep. fin. EDEERdeg.

Example (KLM)

The theory of the random $(2, 4)$-hypergraph has sep. fin. EDEERdeg (by the above remark) but not EDEERP.

KPT context

Let M be a first order structure and Σ^M the universal Aut(M)-ambit.

Proposition (K)

The properties WDERP, DEERP, EDERP of Σ^M are all equivalent to the embedding Ramsey property of M which is equivalent to extreme amenability of Aut(M).

However, sep. fin. EDERdeg of Σ^M is weaker than the property of M of having finite embedding Ramsey degree.

Theorem (K)

If M has finite embedding Ramsey degree, then Σ^M has sep. fin. EDERdeg, and so the Ellis group of Σ^M is profinite which implies that any minimal distal Aut(M)-flow is profinite.